MANIFOLDIC HOMOLOGY AND CHERN-SIMONS FORMALISM

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INTRODUCTION

Manifoldic homology (we suggest this term instead of "topological chiral homology with constant coefficients" from [Lur]) is a far-reaching generalization of Hochschild homology. In the theory of Hochschild and cyclic homology the additive Dennis trace map (e. g. [Lod, 8.4.16]) plays an important role. Let A be an associative algebra. Denote by $L(A)$ the underlying Lie algebra of A. Then the additive Dennis trace gives a map from $H_*(L(A))$ to $HH_*(A)$. The aim of the present note is to generalize this morphism to any e_n-algebra A.

Let e_n be the operad of rational chains of the operad of little discs and A be an algebra over it. The complex $A[n-1]$ is equipped with homotopy Lie algebra structure, denote it by $L(A)$. Fix a compact oriented n-manifold M without borders. For simplicity we restrict ourselves to parallelizable manifolds, this restriction may be removed by introducing framed little discs as in [Sal]. Denote by $HM_*(A)$ the manifoldic homology of A on M introduced in Definition 1 below, and by $H_*(L(A))$ the Lie algebra homology. Then Proposition 3 gives a morphism $H_*(L(A)) \to HM_*(A)$.

In [AS], [BC] and other papers the perturbative Chern-Simons invariant was introduced. The perturbative Chern-Simons invariant of a parallelized compact n-manifold M without borders takes value in the cohomology of the graph complex. The graph complex computes cohomology of the Lie algebra of Hamiltonian vector fields on a super-symplectic vector space with a symplectic form of degree $1-n$. Due to the formality theorem for e_n operad ([LV]) the latter Lie algebra is $L(A)$ for some e_n algebra A. We claim that the perturbative Chern-Simons invariant is the composition of our morphism $H_*(L(A)) \to HM_*(A)$ with an augmentation $HM_*(A) \to \mathbb{Q}$. We hope to develop this statement elsewhere.

Another interesting directions for future research is to generalize our morphism to n-categories (is it connected with the quantum Chern-Simons invariant?) and to interpret it in topological terms (in the context of the Goodwillie calculus?).

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1. TREES AND L_∞

A tree is an oriented connected graph with three type of vertices: *root* has one incoming edge and no outgoing ones, *leaves* have one outgoing edge and no incoming ones and *internal vertexes* have one outgoing edge and more than one

incoming ones. Edges incident to leaves will be called *inputs*, the edge incident to the root will be called the *output* and all other edges will be called *internal edges*. The degenerate tree has one edge and no internal vertexes. Denote by $T_k(S)$ the set of non-degenerate trees with k internal edges and leaves labeled by a set S.

For two trees $t_1 \in T_{k_1}(S_1)$ and $t_2 \in T_{k_2}(S_2)$ and an element $s \in S_1$ the composition of trees $t_1 \circ_s t_2 \in T_{k_1+k_2+1}$ is obtained by identification of the input of t_1 corresponding to s and the output of t_2. Composition of trees is associative and the degenerate tree is the unit. The set of trees with respect to the composition forms an operad.

Call the tree with only one internal vertex the *star*. Any non-degenerate tree with k internal edges may be uniquely presented as a composition of $k + 1$ stars.

The operation of *edge splitting* is the following: take a non-degenerate tree, present it as a composition of stars and replace one star with a tree that is a product of two stars and has the same set of inputs . The operation of an edge splitting depends on a internal vertex and a proper subset of incoming edges.

For a non-degenerate tree t denote by $\mathrm{Det}(t)$ the one-dimensional $\mathbb{Q}$-vector space that is the determinant of the vector space generated by internal edges. For $s > 1$ consider the complex

$$(1) \qquad L(s)\colon \bigoplus_{t \in T_0(\mathbf{s})} \mathrm{Det}(t) \to \bigoplus_{t \in T_1(\mathbf{s})} \mathrm{Det}(t) \to \bigoplus_{t \in T_2(\mathbf{s})} \mathrm{Det}(t) \to \cdots$$

the cohomological degree of a tree $t \in T_k(\mathbf{s})$ is $2 - s + k$, differential is given by all possible splitting of an edge (see e. g. [GK]). The composition of trees equips the sequence $L(i) \otimes \mathrm{sgn}$ with the structure of a *dg*-operad, here sgn is the sign representation of the symmetric group.

This operad is called L_∞ *operad*. Denote by $L_\infty[n]$ the *dg*-operad given by $L(s)[n(s-1)] \otimes (\mathrm{sgn})^n$.

2. Fulton-MacPherson operad

The Fulton-MacPherson compactification is introduced in [GJ] and [Mar], see also [Sal] and [AS]. We cite here its properties that are essential for our purposes.

For a finite set S denote by $(\mathbb{R}^n)^S$ the set of ordered S-tuples in $\mathbb{R}^n$. For a finite set S denote by $\Delta_S\colon \mathbb{R}^n \to (\mathbb{R}^n)^S$ the diagonal embedding. We will denote by $\mathbf{n}$ the set of n elements.

Let $\mathscr{C}_S^0(\mathbb{R}^n) \subset (\mathbb{R}^n)^S$ be the space of ordered pairwise distinct points in $\mathbb{R}^n$ labeled by S. The *Fulton-MacPherson compactification* $\mathscr{C}_S(\mathbb{R}^n)$ is a manifold with corners with interior $\mathscr{C}_S^0(\mathbb{R}^n)$. The projection $\mathscr{C}_S(\mathbb{R}^n) \xrightarrow{\pi} (\mathbb{R}^n)^S$ is defined, which is an isomorphism on $\mathscr{C}_S^0(\mathbb{R}^n) \subset \mathscr{C}_S(\mathbb{R}^n)$. Moreover there is a sequence of manifolds with corners $F_n(S)$ labeled by finite sets and maps $\phi_{S_1,\ldots,S_k}$ that fit in the diagram

$$
\begin{array}{ccc}
F_n(S_1) \times \cdots \times F_n(S_k) \times \mathscr{C}_{\mathbf{k}}(\mathbb{R}^n) & \xrightarrow{\phi_{S_1,\ldots,S_k}} & \mathscr{C}_{(S_1 \cup \cdots \cup S_k)}(\mathbb{R}^n) \\
\downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} \\
(\mathbb{R}^n)^k & \xrightarrow{\Delta_{S_1} \times \cdots \times \Delta_{S_k}} & (\mathbb{R}^n)^{(S_1 \cup \cdots \cup S_k)}
\end{array}
$$

where the left arrow is the projection to the point on the first factors and π on the last one. Restrictions of $\phi_{S_1,\ldots,S_k}$ to $F_n(S_1) \times \cdots \times F_n(S_k) \times \mathscr{C}_{\mathbf{k}}^0(\mathbb{R}^n)$ are isomorphisms onto the image. It follows that $F_n(S) = \pi^{-1}\vec{0}$, where $\vec{0} \in (\mathbb{R}^n)^S$ is S-tuple

sitting at the origin. Being restricted on $F_n(S) \subset \mathscr{C}_S(\mathbb{R}^n)$, maps ϕ equip $F_n(S)$ with an operad structure:

$$\phi_{\mathbf{s_1},\dots,\mathbf{s_k}} \colon F_n(\mathbf{s_1}) \times \cdots \times F_n(\mathbf{s_k}) \times F_n(\mathbf{k}) \to F_n(\mathbf{s_1} + \cdots + \mathbf{s_k})$$

Manifolds $\mathscr{C}_{\mathbf{k}}(\mathbb{R}^n)$ and $F_n(\mathbf{k})$ are equipped with a k-th symmetric group action consistent with its natural action on $\mathscr{C}^0_{\mathbf{k}}(\mathbb{R}^n)$ and all maps are compatible with this action. After [Sal], [GJ], [Mar] call it the *Fulton-MacPherson operad*.

There is a map of sets $F_n(S) \xrightarrow{\mu} T(S)$ that subdivides $F_n(S)$ into smooth strata. Preimage of a tree from $T_k(S)$ is a smooth submanifold of codimension k in F_n and of codimension $k+1$ in $\mathscr{C}_S(\mathbb{R}^n)$. Map μ is consistent with the operad structure in the sense that the preimage of a composition is composition of preimages. Strata of codimension one of $\mathscr{C}_S(\mathbb{R}^n)$ correspond to the star with leaves labeled by $S' \subset S$. These are blow-ups of the corresponding diagonal minus pull backs of smaller diagonals. These strata freely generate the Fulton-MacPherson operad as a set.

Denote by $C_*(F_n)$ the *dg*-operad of rational chains of F_n. For a tree $t \in T(S)$ let $[\mu^{-1}(t)] \in C_*(F_n(S))$ be the chain presented by its preimage under μ.

Proposition 1. *Map $[\mu^{-1}(\cdot)]$ gives a morphism from complex $L(s)[s(1-n)]$ (see (1)) to $C_*(F_n(\mathbf{s}))$.*

Proof. To show that the map commutes with the differential, one should note that two strata given by μ with dimensions differing by 1 are incident if and only if one of the corresponding trees is obtained from another by edge splitting. To control signs one should note that the conormal bundle to a stratum labeled by a tree has a basis labeled by internal edges of the tree. $\square$

It follows that there is a morphism of *dg*-operads

$$(2) \qquad\qquad L_\infty[1-n] \to C_*(F_n)$$

Let e_n be the *dg*-operad of rational chains of the operad of little n-discs.

Proposition 2. *Operad $C_*(F_n)$ is weakly homotopy equivalent to e_n.*

Proof. See [Sal, Proposition 3.9]. $\square$

Thus there is a homotopy morphism of operads $L_\infty[1-n] \to e_n$.

3. Manifoldic homology

Let M be an n-dimensional parallelized compact manifold without borders. In the same way as for $\mathbb{R}^n$ there is the Fulton-MacPherson compactification $\mathscr{C}_S(M)$ of the space $\mathscr{C}^0_S(M)$ of ordered pairwise distinct points in M labeled by S; inclusion $\mathscr{C}^0_S(M) \hookrightarrow \mathscr{C}_S(M)$ is a homotopy equivalence. There is a projection $\mathscr{C}_S(M) \xrightarrow{\pi} M^S$ and maps $\phi_{S_1,\dots,S_k}$ that fit in the diagram

$$
\begin{array}{ccc}
F_n(S_1) \times \cdots \times F_n(S_k) \times \mathscr{C}_{\mathbf{k}}(M) & \xrightarrow{\phi_{S_1,\dots,S_k}} & \mathscr{C}_{(S_1 \cup \cdots \cup S_k)}(M) \\
\downarrow{\scriptstyle\pi} & & \downarrow{\scriptstyle\pi} \\
M^k & \xrightarrow{\Delta_{S_1} \times \cdots \times \Delta_{S_k}} & M^{(S_1 \cup \cdots \cup S_k)}
\end{array}
$$

and are isomorphisms on $F_n(S_1) \times \cdots \times F_n(S_k) \times \mathscr{C}^0_{\mathbf{k}}(M)$, where $\Delta_S \colon M \to M^S$ are the diagonal maps. It follows that spaces $\mathscr{C}_*(M)$ form a right module over the

PROP generated by the Fulton-MacPherson operad

$$P(F_n)(m,l) = \bigcup_{\sum m_i = m} F_n(m_1) \times \cdots \times F_n(m_l).$$

This module as a set is freely generated by $\mathscr{C}_*^0(M)$. The stratification on F_n defines a stratification on $\mathscr{C}_*(M)$.

Denote by $C_*(\mathscr{C}_{\mathbf{k}}(M))$ the complex of rational chains of the Fulton-MacPherson compactification.

Definition 1. For a $C_*(F_n)$-algebra A and a compact parallelized n-manifold without borders M call the complex $CM_*(A) = C_*(\mathscr{C}_*(M)) \otimes_{C_*(P(F_n))} A$ the *manifoldic chain complex* of A on M. Call the homology of the manifoldic chain complex the *manifoldic homology* of A on M.

This definition is based on Definition 4.14 from [Sal]. By Proposition 2 one may pass from a $C_*(F_n)$-algebra to an e_n-algebra. As it is shown in [Lur], the manifoldic homology is the same as the topological chiral homology with constant coefficients introduced in *loc. cit* of this e_n-algebra.

Let $(\mathfrak{g}, d)$ be a complex which is an L_∞-algebra. Let $l_{i>1} \colon \Lambda^i \mathfrak{g}[i-2] \to \mathfrak{g}$ be its higher brackets, that is, the operations in complex (1) corresponding to the star trees. The structure of L_∞-algebra may be encoded in a derivation $D = D_1 + D_2 + \ldots$ on the free super-commutative algebra generated by $\mathfrak{g}^\vee[1]$, where D_1 is dual to d and D_i is dual to l_i on generators and are continued on the whole algebra by the Leibniz rule. The *Chevalley-Eilenberg chain complex* $CE_*(\mathfrak{g})$ is the super-symmetric power $S^*(\mathfrak{g}[-1])$ with the differential $\mathrm{d}_{\mathrm{tot}} = d + \theta_2 + \theta_3 + \ldots$, where θ_i is dual to D_i.

Denote by $[\mathscr{C}_{\mathbf{k}}^0] \in C_*(\mathscr{C}_{\mathbf{k}}(M))$ the chain given by the submanifold $\mathscr{C}_{\mathbf{k}}^0(M)$ in $\mathscr{C}_{\mathbf{k}}(M)$. For a $C_*(F_n)$-algebra A and a cycle $c \in C_*(\mathscr{C}_{\mathbf{k}}(M))$ denote by $(a_1 \otimes \cdots \otimes a_k) \otimes_{\Sigma_k} c \in CM_*(A)$ the chain given by the tensor product over the symmetric group. Recall that by (2) for any $C_*(F_n)$-algebra A the complex $A[n-1]$ is equipped with a L_∞ structure. Denote this L_∞-algebra by $L(A)$. Denote by $\mathrm{Alt}(a_1 \otimes \cdots \otimes a_k)$ the sum $\sum_\sigma \pm a_{\sigma(1)} \otimes \cdots \otimes a_{\sigma(k)}$ by all permutations, where signs are sign given by the sign of the permutation and the Koszul sign rule.

Proposition 3. *For a $C_*(F_n)$-algebra A and a parallelized compact manifold without borders M the map* $\mathrm{T} \colon a_1 \wedge \cdots \wedge a_k \mapsto \mathrm{Alt}(a_1 \otimes \cdots \otimes a_k) \otimes_{\Sigma_k} [\mathscr{C}_{\mathbf{k}}^0]$ *defines a morphism from Chevalley-Eilenberg complex $CE_*(L(A))$ to the manifoldic chain complex $CM_*(A)$.*

Proof. Denote the total differentials on both complexes $CE_*(L(A))$ and $CM_*(A)$ by $\mathrm{d}_{\mathrm{tot}}$. One needs to show that $\mathrm{d}_{\mathrm{tot}} \circ \mathrm{T} = \mathrm{T} \circ \mathrm{d}_{\mathrm{tot}}$.

The border of $[\mathscr{C}_{\mathbf{k}}^0]$ in $C_*(\mathscr{C}_{\mathbf{k}}(M))$ is the sum of all codimension one strata: $\partial[\mathscr{C}_{\mathbf{k}}^0] = \sum_i \theta_i [\mathscr{C}_{\mathbf{k}-i+1}^0]$. Here θ_i is the symmetrization in $C_*(P(F_n))$ of the operation in $C_*(F_n)$ that corresponds by Proposition 1 to the star with i inputs. This means that

$$\mathrm{d}_{\mathrm{tot}} \circ \mathrm{T}\,(a_1 \wedge \cdots \wedge a_k) = (d\,\mathrm{Alt}(a_1 \otimes \cdots \otimes a_k)) \underset{\Sigma_k}{\otimes} [\mathscr{C}_{\mathbf{k}}^0] + \mathrm{Alt}(a_1 \otimes \cdots \otimes a_k) \underset{\Sigma_k}{\otimes} \sum_{i>1} \theta_i [\mathscr{C}_{\mathbf{k}-i+1}^0]$$

One may carry θ's from one factor of $\otimes_{\Sigma_k}$ to another by the very definition of the tensor product over $C_*(P(F_n))$. And the action of θ's on the alternating sum again by definition is given by the higher brackets of the L_∞-algebra. After summing

with d it gives the differential on the Chevalley-Eilenberg complex. It follows that $d_{tot} \circ T = T \circ d_{tot}$. $\qquad \square$

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